Section 1 Unesco

الجامعة اللبنانية
UNIVERSITE LIBANAISE

...ية الإعلام
الفرع الأول

1st Semester
Instructor : Dr. Abbas Rammal
Duration : 90 minutes

Course of Mathematics
Calculus

Exercise 1 . (10 points) Find the derivative of each of the following functions :

1. $f(x) = x\ln(e^x + 1)$.
2. $g(x) = \arctan(\frac{x+1}{x})$.

Exercise 2 . (15 points) Let f be the real function defined by :

$$f(x) = \begin{cases} x + a + \sqrt{x^2 + x + 1} & if \quad x < -1 \\ ax - b + a & if \quad -1 \leqslant x \leqslant 1 \\ bx^2 - 2x & if \quad x > 1 \end{cases}$$

where a and $b \in \mathbb{R}$.
Determine a and b so that f be continuous at $x = -1$ and $x = 1$.

Exercise 3 . (15 points) Let f be the function defined on $I =]0, +\infty[$ by $f(t) = e^{\frac{1}{t}}$.

1. Show that $f'(t) = \frac{-1}{t^2}e^{\frac{1}{t}}$, $\forall\, t \in I$.
2. Let $x > 0$. By applying the Mean Value Theorem to the function $f(t)$ on $[x, x+1]$, show that

$$\frac{1}{(x+1)^2}e^{\frac{1}{x+1}} < e^{\frac{1}{x}} - e^{\frac{1}{x+1}} < \frac{1}{x^2}e^{\frac{1}{x}}.$$

Exercise 4. **(20 points)**

1. Calculate

$$I = \int \frac{x+4}{x^2 + 2x + 5}\,dx$$

2. Determine the real constants A, B and C such that

$$\frac{1}{(1+t^2)(1+t)} = \frac{A}{1+t} + \frac{Bt + C}{1+t^2}$$

3. Calculate

$$J = \int \frac{1}{(1+t^2)(1+t)}\,dt$$

$$K = \int \frac{\sin x}{(1+\cos^2 x)(1+\cos x)}\,dt$$

Ex1:

$f(x) = x\ln(e^x+1)$ $\qquad (u.v)' = u'v + v'u$

$f'(x) = 1\ln(e^x+1) + \frac{e^x}{e^x+1} . x$ $\qquad (\ln(e^x+1))' = \frac{e^x}{e^x+1}$

$f'(x) = \frac{xe^x}{e^x+1} + \ln(e^x+1)$

$g(x) = \arctan\left(\frac{x+1}{x}\right)$ $\qquad (\arctan(u(x)))' = \frac{u'(x)}{1+(u(x))^2}$

$= \frac{-\frac{1}{x^2}}{1+\left(\frac{x+1}{x}\right)^2} = \frac{-\frac{1}{x^2}}{1+\frac{x^2+2x+1}{x^2}}$ $\qquad \left(\frac{x+1}{x}\right)' = \frac{1.x - 1.(x+1)}{x^2}$

$= \frac{-\frac{1}{x^2}}{\frac{x^2+x^2+2x+1}{x^2}}$ $\qquad = x - x - 1 = -\frac{1}{x^2}$

$g'(x) = -\frac{1}{2x^2+2x+1}$

Ex2:

$$f(x) = \begin{cases} x+a+\sqrt{x^2+x+1} & \text{if } x<-1 \\ ax-b+a & \text{if } -1\leq x\leq 1 \\ bx^2-2x & \text{if } x>1 \end{cases}$$

a, b ? if f continuous on $x=-1$ and $x=1$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x+a+\sqrt{x^2+x+1} = -1+a+\sqrt{1-1+1} = -1+a+\sqrt{1}$
$= -1+a+1 = a$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} ax-b+a = -a-b+a = -b$

$\Rightarrow a = -b$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} ax - b + a = a - b + a = 2a - b$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} bx^2 - 2x = b - 2$

$\Rightarrow 2a - b = b - 2$

$a = -b$ $\quad \begin{cases} a + b = 0 \\ 2a - b - b - 2 = 0 \end{cases} \Rightarrow \begin{cases} a + b = 0 \quad (\times 2) \\ 2a - 2b = 2 \end{cases} \Rightarrow \begin{cases} 2a + 2b = 0 \\ 2a - 2b = 2 \end{cases}$

$2a - b = b - 2$

$4a = 2 \Rightarrow a = \frac{1}{2}$

$\Rightarrow b = -\frac{1}{2}$

$a = \frac{1}{2}$, $b = -\frac{1}{2}$

Ex3:

f defined on $I =]0; +\infty[$

$f(t) = e^{\frac{1}{t}}$

1) $f'(t) = \frac{-1}{t^2} e^{\frac{1}{t}}$?

$f(t) = e^{\frac{1}{t}}$ $\qquad (e^{u(x)})' = u'(x) \cdot e^{u(x)}$

$f'(t) = \frac{-1}{t^2} e^{t}$ $\qquad \left(\frac{1}{t}\right)' = \frac{0 - 1 \cdot 1}{t^2} = \frac{-1}{t^2}$

2) $x > 0$

$f(t)$ on $[x, x+1]$

$\frac{1}{(x+1)^2} e^{\frac{1}{x+1}} < e^{\frac{1}{x}} - e^{\frac{1}{x+1}} < \frac{1}{x^2} e^{\frac{1}{x}}$??

f continuous on $[x, x+1]$ when $x>0$

and f differentiable on $]x, x+1[$ when $x>0$

then $\exists c \in]x, x+1[$ such that

$$f'(c) = \frac{f(x+1) - f(x)}{x+1-x} = \frac{e^{\frac{1}{x+1}} - e^{\frac{1}{x}}}{1} = e^{\frac{1}{x+1}} - e^{\frac{1}{x}}.$$

$x < c < x+1$ when $x>0$

$x^2 < c^2 < (x+1)^2$

$$\frac{1}{(x+1)^2} < \frac{1}{c^2} < \frac{1}{x^2}$$

$$\frac{1}{(x+1)^2} e^{\frac{1}{x+1}} < \underbrace{\frac{1}{c^2} e^{\frac{1}{c}}}_{=-f'(c)} < \frac{1}{x^2} e^{\frac{1}{x}}$$

($e^{\frac{1}{c}} > 0$ because $x < c < x+1$ and $x>0$)

$$\cancel{f'(c) = \frac{1}{c^2} e^{\frac{1}{c}}}$$

$$f'(c) = -\frac{1}{c^2} e^{\frac{1}{c}} \qquad / \; f'(t) = \frac{-1}{t^2} e^{\frac{1}{t}}$$

$$f'(c) = e^{\frac{1}{x+1}} - e^{\frac{1}{x}} \Rightarrow -f'(c) = e^{\frac{1}{x}} - e^{\frac{1}{x+1}}$$

$$\Rightarrow \frac{1}{(x+1)^2} e^{\frac{1}{x+1}} < -f'(c) < \frac{1}{x^2} e^{\frac{1}{x}}$$

$$\Rightarrow \frac{1}{(x+1)^2} e^{\frac{1}{x+1}} < e^{\frac{1}{x}} - e^{\frac{1}{x+1}} < \frac{1}{x^2} e^{\frac{1}{x}}.$$

Exercice 2:

$$I = \int \frac{x+4}{x^2+2x+5}\,dx$$

1) $\int \frac{x+4}{x^2+2x+5}\,dx$

$$= \frac{1}{2}\int \frac{2(x+4)}{x^2+2x+5}\,dx$$

$$= \frac{1}{2}\int \frac{2x+8}{x^2+2x+5}\,dx$$

$$= \frac{1}{2}\left[\int \frac{2x+2}{x^2+2x+5}\,dx + 6\int \frac{1}{x^2+2x+5}\,dx\right]$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx + 3\int \frac{1}{x^2+2x+5}\,dx$$

$\int \frac{2x+2}{x^2+2x+5}\,dx$ $\quad t = x^2+2x+5 \quad dt = (2x+2)\,dx$

$$\to = \int \frac{dt}{t} = \ln|t| = \ln|x^2+2x+5|$$

$$\int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx = \int \frac{1}{(x+1)^2+(2)^2}\,dx$$

$$= \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + c$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx + 3\int \frac{1}{x^2+2x+5}\,dx$$

$$= \frac{1}{2}\ln|x^2+2x+5| + 3\left(\frac{1}{2}\arctan\left(\frac{x+1}{2}\right)\right) + c$$

$$= \frac{1}{2}\ln|x^2+2x+5| + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + c.$$

2) $\frac{1}{(1+t^2)(1+t)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2}$

$\frac{A}{1+t} + \frac{Bt+C}{1+t^2} = \frac{A(1+t^2)+(Bt+C)(1+t)}{(1+t)(1+t^2)} = \frac{A+At^2+Bt+Bt^2+C+Ct}{(1+t)(1+t^2)}$

$= \frac{(A+B)t^2+(B+C)t+(A+C)}{(1+t)(1+t^2)}$

$\frac{1}{(1+t^2)(1+t)} = \frac{(A+B)t^2+(B+C)t+(A+C)}{(1+t^2)(1+t)}$

$\begin{cases} A+B=0 \longrightarrow B=-A \\ B+C=0 \\ A+C=1 \end{cases}$

$-A+C=0$
$A+C=1$
$2C=1 \Rightarrow C=\frac{1}{2} \Rightarrow \boxed{C=\frac{1}{2}}$

$A+C=1 \Rightarrow \boxed{A=\frac{1}{2}}$

$A+B=0 \Rightarrow \boxed{B=-\frac{1}{2}}$

$\frac{1}{(1+t^2)(1+t)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2}$

$= \frac{1}{2}\left(\frac{1}{1+t}\right) + \frac{-\frac{1}{2}t+\frac{1}{2}}{1+t^2} = \frac{1}{2(1+t)} + \frac{-t+1}{2(1+t^2)} = \frac{1}{2(1+t)} + \frac{1-t}{2(1+t^2)}$

3) $J = \int \frac{1}{(1+t^2)(1+t)}\,dt = \frac{1}{2}\int \frac{1}{1+t}\,dt + \frac{1}{2}\int \frac{1-t}{1+t^2}\,dt$

$= \frac{1}{2}\ln|1+t| + \frac{1}{2}\int \frac{dt}{1+t^2} - \frac{1}{2}\int \frac{t\,dt}{1+t^2}$

$= \frac{1}{2}\ln|1+t| + \frac{1}{2}\int \frac{dt}{1+t^2} - \frac{1}{2}\cdot\frac{1}{2}\int \frac{2t\,dt}{1+t^2}$

~~$= \frac{1}{2}\ln|1+t| - \frac{1}{4}$~~

$= \frac{1}{2}\ln|1+t| + \frac{1}{2}\int \frac{dt}{1+t^2} - \frac{1}{4}\ln|1+t^2|$

$$\int \frac{dt}{1+t^2} = \arctan t$$

$$\Rightarrow \int \frac{1}{(1+t^2)(1+t)}\,dt = \frac{1}{2}\ln|1+t| + \frac{1}{2}\arctan t - \frac{1}{4}\ln|1+t^2| + c$$

$$K = \int \frac{\sin x}{(1+\cos^2 x)(1+\cos x)}\,dx \quad ??$$

$$\int \frac{\sin x}{(1+\cos^2 x)(1+\cos x)}\,dx \qquad \int \frac{1}{(1+t^2)(1+t)}\,dt$$

$$\int \frac{\sin x}{(1+\cos^2 x)(1+\cos x)}\,dx \qquad u = \cos x \qquad du = -\sin x\,dx$$

$$= -\int \frac{-\sin x\,dx}{(1+\cos^2 x)(1+\cos x)} = -\int \frac{du}{(1+u^2)(1+u)} = -J(u)$$

$$= -\left[\frac{1}{2}\ln|1+u| + \frac{1}{2}\arctan u - \frac{1}{4}\ln|1+u^2| + c\right]$$

$$= -\frac{1}{2}\ln|1+u| - \frac{1}{2}\arctan u + \frac{1}{4}\ln|1+u^2| + c$$

$$= -\frac{1}{2}\ln|1+\cos x| - \frac{1}{2}\arctan(\cos x) + \frac{1}{4}\ln|1+\cos^2 x| + c$$